IMPLICATION ALGEBRAS

by

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CHAPTER 0

INTRODUCTION AND PRELIMINARIES

In this thesis we present the implication algebras of Abbott [1]. These are algebras of type $(2,0)$ such that the operations and axioms model implication and truth in classical logic. In these structures the interval above each element is a Boolean algebra. We then present the generalization of this by Hardegree [4] and Godowski [3]. In the Hardegree system the intervals above each element are orthomodular lattices and in the Godowski system such intervals are orthomodular posets.

We then develop the generalized orthomodular lattices of Janowitz [5]. We show that dual generalized orthomodular lattices correspond to the Hardegree system.

0.1 Definition. An Orthocomplemented Poset is a poset P, with a least and greatest element 0 and 1 respectively, which admits a mapping $': P \to P$ such that for all $x,y \in P$ we have the following:

(i) $x \leq y \Rightarrow y' \leq x'$

(ii) $x'' =_{\text{def}} (x')' = x$

(iii) x' is complement of x.

0.2 Definition. An Orthomodular Poset (OMP) is an orthocomplemented poset P such that

(i) $a \leq b' \Rightarrow a \vee b$ exists in P, and

(ii) $a \leq b$ and $a \vee b' = 1 \Rightarrow a = b$.

0.3 Definition. An Orthomodular Lattice (OML) is an orthomodular poset which is a lattice.

0.4 Definition. Let $(P,')$ be an orthomodular poset. For $a,b \in P$ we say a is compatible with b, $a \mathrel{C} b$, if there exist mutually orthogonal elements $a_1, b_1, c \in P$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$.

0.5 Theorem. [6, pp. 20-23]. Let L be an orthomodular lattice. These are equivalent.

1) $a \mathrel{C} b$

2) $a = (a \wedge b) \vee (a \wedge b')$

3) $a \mathrel{C} b'$

4) $b \mathrel{C} a$

5) $(a \vee b') \wedge b = a \wedge b$.

One of the most important tools in orthomodular lattice theory is the following:

0.6 Theorem. (Foulis-Holland) [6, p. 25]. Let L be an orthomodular lattice. The sublattice generated by three elements, one of which commutes with the other two, is distributive.

CHAPTER ONE

IMPLICATION ALGEBRAS

In this chapter, we give the definition of an implication algebra, introduced by J. C. Abbott in [1], and prove some basic results.

1.1 Definition. An Implication Algebra (IA) is a pair $\langle I,\cdot\rangle$, where I is a set, and "$\cdot$" is a binary operation on I, satisfying, for all $a,b,c \in I$,

A1) $(a\cdot b)\cdot a = a$, (contraction);

A2) $(a\cdot b)\cdot b = (b\cdot a)\cdot a$, (quasi-commutative);

A3) $a\cdot(b\cdot c) = b\cdot(a\cdot c)$, (exchange).

Henceforth we write ab for $a\cdot b$.

1.2 Example 1. One can show that if B is a Boolean algebra and ab is defined to be $a'\vee b$, then $\langle B,\cdot\rangle$ is an IA. Similarly we may use $ab = a'\wedge b$, and again obtain an IA. Thus every Boolean algebra $\langle I,\vee,\wedge,',0,1\rangle$ determines two implication algebras.

(1) $\langle I,\rightarrow\rangle$ where $a \rightarrow b \equiv a'\vee b$, (Implication algebra),

(2) $\langle I,-\rangle$ where $b - a \equiv a'\wedge b$, (Subtraction algebra).

1.3 Lemma. For any two elements a,b in an implication algebra I we have the following:

(i) $a(ab) = ab$

(ii) $aa = (ab)(ab)$

Proof.

(i) $a(ab) = [(ab)a](ab) = ab.$

(ii) $aa = [(ab)a]a = [a(ab)](ab) = (ab)(ab).$

1.4 Theorem. There exists an element $1 \in I$ such that for all $a \in I$.

(i) $aa = 1;$

(ii) $1a = a;$

(iii) $a1 = 1.$

Proof.

(i) $aa = (ab)(ab) = [(ab)b][(ab)b] = [(ba)a][(ba)a]$

$= (ba)(ba) = bb.$

Hence I contains a constant $1 \equiv aa$ independent of a.

(ii) $1a = (aa)a = a.$

(iii) $a1 = a(aa) = aa = 1.$

1.5 Theorem. For any two elements a,b in an implication algebra I we have

1) $a(ba) = 1;$

2) $[(ab)b]b = ab$;

3) $[(ab)b]a = ba$;

4) $a[(ab)b] = b[(ab)b] = 1$;

5) $(ab)(ba) = ba$.

Proof.

1) $a(ba) = b(aa) = b1 = 1$.

2) $[(ab)b]b = [b(ab)](ab) = 1(ab) = ab$.

3) $[(ab)b]a = [(ba)a]a = [a(ba)](ba) = 1(ba) = ba$.

4) $a[(ab)b] = a[(ba)a] = 1$, $b[(ab)b] = 1$.

5) $(ab)(ba) = b[(ab)a] = ba$.

1.6 Corollary. For a,b in an implication algebra I, we have the following:

(i) $ab = ba$ iff $a = b$, (anti-commutative);

(ii) $ab = b$ iff $ba = a$;

(iii) $ab = a$ iff $a = 1$.

Proof.

The proof follows immediately from the preceding Theorem.

1.7 Theorem. For any three elements a,b,c in an implication algebra I we have the following:

1) $(ac)(bc) = (ca)(ba)$;

2) $[(ab)c]c = [(ab)c](ac)$;

3) $(ab)\{[(ac)b]b\} = 1$;

4) $cb = 1 \Longrightarrow ab = [(ac)b]b$;

5) $a(bc) = (ab)(ac)$, (quasi-distributive);

6) $[(ab)c]c = a[(bc)c]$, (quasi-associative);

7) $\{[(ab)b]c\}c = \{[(cb)b]a\}a$;

8) $ab = 1 \Longrightarrow (bc)(ac) = 1$.

Proof.

1) $(ac)(bc) = b[(ac)c] = b[(ca)a] = (ca)(ba)$;

2) $[(ab)c]c = [c(ab)](ab) = [c(ab)][a(ab)] = [(ab)c](ac)$;

3) $(ab)\{[(ac)b]b\} = (ab)\{[(ac)b](ab)\} = 1$;

4) (i) $(ab)\{[(ac)b]b\} = 1$ by 3;

(ii) $\{[(ac)b]b\}(ab) = a\langle\{[(ac)b]b\}b\rangle$

$= a[(ac)b] = (ac)(ab) = (ac)[1(ab)]$

$= (ac)[(cb)(ab)] = (ac)[(bc)(ac)] = 1$

5) (i) $[a(bc)][(ab)(ac)] = (ab)\{[a(bc)](ac)\}$

$= (ab)\{[b(ac)](ac)\}$

$= (ab)\{[(ac)b]b\} = 1$ by 3;

(ii) $[(ab)(ac)][a(bc)] = [(ab)(ac)][b(ac)]$

$= [(ac)(ab)][b(ab)] = 1$;

6) $[(ab)c]c = [c(ab)](ab) = [a(cb)](ab) = a[(cb)b]$
$= a[(bc)c];$

7) $\{[(ab)b]c\}c = (ab)[(bc)c]$ by 6;
$= (ab)[(cb)b] = (cb)[(ab)b]$
$= (cb)[(ba)a] = \{[(cb)b]a\}a;$

8) $(bc)(ac) = (cb)(ab) = (cb)1 = 1.$

1.8 Theorem. Let $(I,\cdot)$ be a set $\cdot$ satisfies A1,A2 and the quasi distributive law [1.7 (5)], then I is an implication algebra.

Proof.

We need to show that the exchange axiom (A3) holds. Applying 1.7(8) to $b(ab) = 1$ gives

$$[(ab)(ac)][b(ac)] = 1 \quad \text{so} \quad [a(bc)][b(ac)] = 1.$$

Similarly, $[b(ac)][a(bc)] = 1$. So the result follows from the anti-commutative law.

1.9 Theorem. Every implication algebra $\langle I,\cdot\rangle$ determines a poset $\langle I,\leq,1\rangle$ with greatest element 1 under $a \leq b \Longleftrightarrow ab = 1$.

Proof.

(i) $aa = 1 \Rightarrow a \leq a$; hence $\leq$ is reflexive.

(ii) $a \leq b$ and $b \leq a \Rightarrow ab = ba = 1$, hence $a = b$.

Thus $\leq$ is anti-symmetric.

(iii) $a \leq b$ and $b \leq c \Rightarrow ab = bc = 1$.

So $ac = a(1c) = a[(bc)c] = a[(cb)b] = (cb)(ab)$

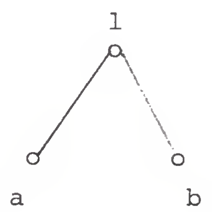
$= (cb)1 = 1$.

Hence $a \leq c$. Thus $\leq$ is transitive.

1.10 Example. Here is an example of an implication algebra which is not a lattice. Let $A = \{1,a,b\}$ and define "$\cdot$" as follows

$\cdot$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

This is the Hasse diagram for the above example



1.11 Theorem. For all a,b,c in an implication algebra I with $a \leq b$ we have $ca \leq cb$ and $bc \leq ac$.

Proof.

(i) $a \leq b \Rightarrow ab = 1$. Hence $(ca)(cb) = c(ab) = 1$.

Thus $ca \leq cb$.

(ii) $(bc)(ac) = (cb)(ab) = 1$. Thus $bc \leq ac$.

1.12 *Theorem*. If a,b are in an implication algebra I, then $a \leq b$ iff $b = xa$, for some $x \in I$.

Proof.

(i) Let $a \leq b$. Then $ab = 1$. So $b = 1b = (ab)b = xa$ where $x = ba$.

(ii) Let $b = xa$ for some $x \in I$. Then $ab = a(xa) = 1$.

Now we define a *join semilattice* to be a poset in which any two elements have a least upper bound.

1.13 *Lemma*. The poset $(I,\cdot,\leq)$ is a join semilattice in which $a \vee b = (ab)b$.

Proof.

The proof is clear.

1.14 *Theorem*. For all a,b,c in an implication algebra I, if $c \leq b$, then $ab = (ac) \vee b$.

Proof.

$$\begin{aligned}(ac) \vee b &= [(ac)b]b = [b(ac)](ac) = [a(bc)](ac) \\ &= a[(bc)c] = a[(cb)b] = a(1b) = ab.\end{aligned}$$

1.15 *Theorem*. For all $a,b,c \in I$. $a(b \vee c) = (ab) \vee c = (ab) \vee (ac)$.

Proof.

$$a(b \vee c) = a[(bc)c] = [(ab)c]c = (ab) \vee c$$
$$= [c(ab)](ab) = [a(cb)](ab) = (ab) \vee (ac).$$

1.16 Corollary. If $c \leq a,b$, then $a \wedge b = [a(bc)]c$ is the greatest lower bound for a and b.

Proof.

The proof follows immediately from preceding theorems.

1.17 Theorem. If $a,c \in I$ with $c \leq a$. Then ac is a complement of a in the principal filter $[c,1]$.

Proof.

[Note: $[c,1] = \{x : c \leq x\} = \{x : x = cy$ for some $y \in I\}$].

We have to show (i) $a \vee ac = 1$ and (ii) $a \wedge ac = c$.

(i) $a \vee ac = [a(ac)](ac) = 1$;

(ii) since $c \leq a$, ac, $a \wedge ac = \{a[(ac)c]\}c = 1c = c$.

1.18 Theorem. For any $a \in I$, the principal filter $[a,1]$ is a Boolean algebra.

Proof.

Let $b,c,d \in [a,1]$. Then it is sufficient to prove the distributive law: $b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d)$. Let $r = (b \wedge c) \vee (b \wedge d)$ and $s = b \wedge c$ then $s \leq r$, so that $bs \leq br$. But since $s \leq b,c$ we can write $s = b \wedge c = [b(cs)]s$.

Hence $bs = b\{[b(cs)]s\} = [b(cs)](bs) = [c(bs)](bs) = bs \vee c = bc$. Therefore $c \leq bc = bs \leq br$. Similarly $d \leq br$, so that $c \vee d \leq br$, i.e., $(c \vee d)(br) = 1$. Finally $r = 1r = [(c \vee d)(br)]r$ $= b \wedge (c \vee d)$, since $r \leq b, (c \vee d)$. Thus $r = (b \wedge c) \vee (b \wedge d)$ $= b \wedge (c \vee d)$.

Hence every implication algebra $\langle I, \cdot \rangle$ determines a poset $\langle I, \leq \rangle$ which is a join semilattice in which every principle filter is a Boolean algebra.

CHAPTER 2

2.1 Quasi Implication Algebras

Quasi Implication Algebras (QIA) are intended to generalize orthomodular lattices in the same way that implication algebras generalize Boolean lattices. Gary M. Hardegree [4] has defined a QIA in the following way:

2.1.1 <u>Definition</u>. A <u>Quasi Implication Algebra</u> is a set Q together with a binary operation "$\cdot$" satisfying the following:

H1) $(a\cdot b)\cdot a = a$;

H2) $(a\cdot b)\cdot(a\cdot c) = (b\cdot a)\cdot(b\cdot c)$;

H3) $[(a\cdot b)\cdot(b\cdot a)]\cdot a = [(b\cdot a)\cdot(a\cdot b)]\cdot b$.

Henceforth $a\cdot b$ is written as ab.

2.1.2 <u>Theorem</u>. Every implication algebra is a quasi implication algebra.

<u>Proof</u>.

Given an IA $\langle I,\cdot\rangle$, we wish to show H1-3 hold.

H1) Is equivalent to A1 in IA.

H2) $(ab)(ac) = a(bc) = b(ac) = (ba)(bc)$.

H3) $[(ab)(ba)]a = \{b[(ab)a]\}a = (ba)a = (ab)b$
$= \{a[(ba)b]\}b = [(ba)(ab)]b$.

2.1.3 <u>Theorem</u>. Let L be an orthomodular lattice and let

"$\cdot$" be the binary operation on L defined by $ab = a' \vee (a \wedge b)$. Then $\langle L, \cdot \rangle$ is a quasi implication algebra.

Proof.

It is sufficient to show that $\langle L, \cdot \rangle$ satisfies axioms H1-3. Here we show

(H1) $(ab)a = (ab)' \vee (ab \wedge a) = [a' \vee (a \wedge b)]' \vee \{[a' \vee (a \wedge b)] \wedge a\}$

$= [a \wedge (a' \vee b')] \vee \{[a' \vee (a \wedge b)] \wedge a\}$

$= [a \wedge (a' \vee b')] \vee (a \wedge b) = a \wedge 1 = a.$

(H2) $(ab)(ac) = (ba)(bc)$:

$(ab)(ac) = (ab)' \vee [(ab) \wedge (ac)] = [a \wedge (a' \vee b')] \vee$

$\vee \{[(a' \vee (a \wedge b)] \wedge [a' \vee (a \wedge c)]\}$

$= [a \wedge (a' \vee b')] \vee [a' \vee (a \wedge b \wedge c)]$

$= \{[(a \wedge (a' \vee b')] \vee a'\} \vee (a \wedge b \wedge c)$

$= [(a \vee a') \wedge (a' \vee b' \vee a')] \vee (a \wedge b \wedge c)$

$= [a' \vee b'] \vee (a \wedge b \wedge c) = (a \wedge b)c.$

By analogous reasoning, one shows $(ba)(bc) = (b \wedge c)c$.

(H3) $[(ab)(bc)]a = [(ba)(ab)]b$:

First of all, $(ab)(ba) = a \vee (a' \wedge b')$. Thus, $[(ab)(ba)]a = [a \vee (a' \wedge b')]a = a \vee b$. By analogous reasoning, one shows $[(ba)(ab)]b = b \vee a$. Thus L is a quasi implication algebra.

2.1.4 Theorem. For every $a,b,c \in Q$ where Q is a quasi implication algebra we have the following:

1) $a(ab) = ab$

2) $aa = (ab)(ab) = bb =_{def} 1$

3) $1a = a$, $a1 = 1$

4) $(ab)(ac) = a[(ab)c] = a[(ab)(ac)]$

5) $ab = ba \Longleftrightarrow a = b$

If now we define $a \leq b$ to mean $ab = 1$, we have

6) $a \leq b \Rightarrow a(bc) = ac$

7) $\langle Q, \leq, 1\rangle$ is a partially ordered set bounded above by 1

8) $b \leq c \Rightarrow ab \leq ac$

9) $a \leq (ab)b$

10) $a \leq b \Longleftrightarrow (ab)b = b$.

Proof.

1) $a(ab) = [(ab)a](ab) = ab$.

2) $aa = [(ab)a]a = \{(ab)[(ab)a]\}a = \{[a(ab)][(ab)a]\}a$

by H3 $= \{[(ab)a][a(ab)]\}(ab) = [a(ab)](ab) = (ab)(ab)$.

3-6 are clear.

7) (i) $a \leq a$ is clear.

(ii) $a \leq b$ and $b \leq a$. Then $ab = ba = 1$ (by 6). Hence by 5 $a = b$.

(iii) $a \leq b$ and $b \leq c$. Then $bc = 1$. $ac = a(bc)$ $= a1 = 1$. Thus $a \leq c$.

(iv) $a \leq 1$ is clear.

8 & 9 are clear.

10. ($\Rightarrow$) Suppose $a \leq b$. Then $ab = 1$. Therefore, $(ab)b = 1b = b$.

($\Leftarrow$) Suppose $(ab)b = b$. Then $ab = a[(ab)b] = 1$ (by 10). Thus, $a \leq b$.

2.1.5 <u>Lemma</u>. Let Q be a quasi implication algebra. Then for all $a,b \in Q$ we have $[(ab)b]a = ba$.

<u>Proof</u>.

$$[(ab)b]a = [(ab)b][(ab)a] = (H2) = [b(ab)](ba) = 1(ba) = ba.$$

2.1.6 <u>Theorem</u>. Let I be a quasi implication algebra satisfying the exchange axiom: $a(bc) = b(ac)$. Then I is an implication algebra.

<u>Proof</u>.

We need to show that the quasi commutative law holds, i.e., $(ab)b = (ba)a$.

$[(ab)b][(ba)a] = (ba)\{[(ab)b]a\} = (ba)(ba) = 1$ by 2.1.5. And similarly $[(ba)a][(ab)b] = 1$. Hence $(ab)b = (ba)a$, and I is an implication algebra.

2.2 Compatibility in Quasi-Implication Algebras

In the theory of orthomodular lattices, one can define a binary relation C of <u>compatibility</u> as follows.

$$\text{(C)} \quad aCb \equiv_{df} a = (a \wedge b) \vee (a \wedge b')$$

This is a restatement of Definition 0.4.
It may be noted that C can be defined on general ortholattices, and the relation C is symmetric on an ortholattice L (aCb iff bCa) if and only if L is orthomodular. It is a further result of Foulis and Holland that C is universal on an ortholattice L (aCb for all $a,b \in L$) if and only if L is Boolean. Thus, what distinguishes Boolean lattices from more general orthomodular lattices is the existence in the latter of a non-trivial compatibility relation.

Now, whereas classical implication algebras correspond to Boolean lattices, quasi-implication algebras are intended to correspond to orthomodular lattices. One might therefore expect the notion of compatibility to generalize to quasi-implication algebras. In particular, we seek a binary relation C definable on general quasi-implication algebras, which has the following properties: (1) In the case of QIA's induced by orthomodular lattices, the implicational compatibility relation coincides with orthomodular compatibility. (2) In the case of general QIA's, every QIA in which every pair of elements is compatible is an IA.

Concerning the criterion (1), we note the following theorem of orthomodular lattices.

$$\text{(T)} \quad a = (a \wedge b) \vee (a \wedge b') \quad \text{iff} \quad a \leq b' \vee (b \wedge a)$$

Proof.

($\Rightarrow$) Suppose $a = (a \wedge b) \vee (a \wedge b')$. Then, since $(a \wedge b) \vee (a \wedge b') \leq b' \vee (b \wedge a)$, $a \leq b' \vee (b \wedge a)$.

($\Leftarrow$) Suppose $a \leq b' \vee (b \wedge a)$. Then $a = a \wedge [b' \vee (b \wedge a)]$. But $(b \wedge a)$ is compatible with both b' and a, so by the Foulis-Holland theorem, $\{(b \wedge a), b', a\}$ is a distributive triple. Therefore, $a \wedge [b' \vee (b \wedge a)] = (a \wedge b') \vee ((a \wedge (b \wedge a)) = (a \wedge b) \vee$ $\vee (a \wedge b')$. Thus, $a = (a \wedge b) \vee (a \wedge b')$.

In light of Theorem (T), and in light of the definition of compatibility and quasi-implication in orthomodular lattices, we see that the following holds in an OML:

(T*) aCb iff $a \leq ba$.

With this in mind, we introduce the following definition.

Definition. aCb $\equiv_{df}$ $a \leq ba$

2.2.1 Lemma. aCa (reflexivity).

Proof.

$a \leq 1 = aa$.

2.2.2 Lemma. aCab.

Proof.

$a \leq a = (Q1) = (ab)a$.

2.2.3 Lemma. aC1.

Proof.

$a \leq a = 1a$.

2.2.4 *Lemma.* $a \leq b$ implies aCb.

Proof.

Suppose $a \leq b$. Then by 2.1.4(b), $a(ba) = aa = 1$. Thus, $a \leq ba$, i.e., aCb.

2.2.5 *Lemma.* $a \leq b$ implies bCa.

Proof.

Suppose $a \leq b$. Then $ab = 1$, so $b \leq ab$, i.e., bCa.

2.2.6 *Lemma.* Let aCb. Then the following hold:

1) $a(bc) = (ab)(ac)$,

2) bCa; (symmetry),

3) $a(bc) = b(ac)$,

4) $(ba)(ab) = ab$.

Proof.

The proof is clear.

2.2.7 *Theorem.* Let Q be a quasi implication algebra. If aCb holds for each $a,b \in Q$, then Q is an implication algebra.

Proof.

A1) Is equivalent to H1.

A2) By Lemma 2.2.6 $ab = (ba)(ab)$ and $ba = (ab)(ba)$. Therefore, $(ab)b = [(ba)(ab)]b = (H3) = [(ab)(ba)]a = = (ba)a$. Thus $(ab)b = (ba)a$.

A3) Is equivalent to 2.2.6(3). Hence Q is an implication algebra.

2.2.8 Lemma. Let Q be a quasi implication algebra. Then for all $a,b \in Q$,

(i) $a \leq (ab)(ba)$.

(ii) If $a \leq b$, then $b = (ba)a$.

Proof.

The proof is clear.

2.2.9 Lemma. $a \leq b$ implies $b(ac) = ac$.

Proof.

Suppose $a \leq b$. Then $ab = 1$. Also by Lemma 2.2.5, bCa. Therefore by Lemma 2.2.6(1), $b(ac) = (ba)(bc) = (H2) = (ab)(ac) = = 1(ac) = ac$.

2.2.10 Theorem. Let Q be a quasi implication algebra. Then

1) $(Q,\leq)$ is a join semi lattice with $a \vee b = [(ab)(ba)]a$.

2) If $b \leq x$, then $x^{\#(b)} \equiv_{def.} xb$ is a complement for x in $Q[b,1]$.

3) $(Q[b,1],\#(b))$ is an orthomodular lattice.

4) If $b \leq a \leq x$, then $x^{\#(a)} = x^{\#(b)} \vee a$.

5) If $x,y \geq b$, then $x \wedge y$ exists in Q.

Proof.

1) By Theorem 2.1.4(7) Q is a poset. Therefore we need only show $a,b \leq [(ab)(ba)]a$ and if $a,b \leq c$, then $[(ab)(ba]a \leq c$. First note that by Lemma 2.2.8(i) $a \leq (ab)(ba)$. Hence by Theorem 2.1.4(4) $a\{[(ab)(ba)]a\} = aa = 1$. Thus $a \leq [(ab)(ba)]a$ and by symmetry and (H3), $b \leq [(ab)(ba)]a$. Now let $a,b \leq c$, then by Lemma 2.2.4 aCc and bCc. Therefore $abCc$ and $a \vee bCc$. Furthermore since $a \leq a \vee b$, by Lemma 2.2.4, $a \vee bCa$. Hence $a \vee bCca$. Now since $a \leq c$, by Lemma 2.2.8(ii), $c = (ca)a$. Thus we have $\{[(ab)(ba)]a\}c = (ca)[(ab)(ba)]$; so in order to show $a \vee b \leq c$, it is sufficient to show $ca \leq (ab)(ba)$. But by Lemma 2.2.8(i) $a \leq (ab)(ba)$; then $ca \leq c[(ab)(ba)]$. So $c[(ab)(ba)] = (ab)(ba)$ and hence $[(ab)(ba)]a \leq c$.

2) (i) $x \vee x^{\#(b)} = x \vee xb = \{[x(xb)][(xb)x]\}x = xx = 1$.

(ii) We delay the proof that $x \wedge x^{\#(b)} = b$, for $x \geq b$, to the next paragraph.

3) (i) $x^{\#(b)\#(b)} = (xb)b = 2.2.8(ii) = x$.

(ii) To show $b \leq x \leq y$ implies $y^{\#(b)} \leq x^{\#(b)}$ we note $xy = 1$, since $x \leq y$. Now $b \leq x$ for all $x \in Q$; so in particular, $b \leq (yx)b$. Therefore, $yb \leq y[(yx)b]$. But $y[(yx)b] = (yx)(yb) = = (xy)(xb) = xb$. Thus $yb \leq xb$, i.e. $y^{\#(b)} \leq x^{\#(b)}$.

(iii) (3i) and (3ii) show that $\#(b)$ is an involution on $Q[b,1]$. It follows that the De Morgan Laws hold in $Q[b,1]$. Therefore, since $\underset{Q[b,1]}{\text{l.u.b}} \{x^{\#(b)}, y^{\#(b)}\}$ exists for $b \leq x,y$, we know that $\underset{Q[b,1]}{\text{g.l.b}} \{x,y\}$ exists also and equals $\underset{Q[b,1]}{\text{l.u.b}} \{x^{\#(b)}, y^{\#(b)}\}^{\#(b)}$. Thus $Q[b,1]$ is an involution lattice. In particular for $b \leq x$, we have $\underset{Q[b,1]}{\text{g.l.b}} \{x, x^{\#(b)}\} = = \underset{Q[b,1]}{\text{l.u.b}} \{x, x^{\#(b)}\}^{\#(b)} = 1^{\#(b)} = b$, which complete the proof of (2ii).

(iv) It is also clear that if $b \leq x \leq y$ and $y \wedge x^{\#(b)} = b$, then $x = y$. Thus $Q[b,1]$ is an OML.

4) Let $b \leq a \leq x$. We want to show $x^{\#(a)} = x^{\#(b)} \vee a$.

(i) Since $b \leq a$, $xb \leq xa$; so $x^{\#(b)} \leq x^{\#(a)}$. Also $a \leq x^{\#(a)}$. Hence $x^{\#(b)} \vee a \leq x^{\#(a)}$.

(ii) $x^{\#(a)}(x^{\#(b)} \vee a) = (xa)(xb \vee a) =$

$= (xa)(\{[a(xb)][(xb)a]\}a) = (ax)\{[(a(xb)][(xb)a)]x\} =$

$= \{[a(xb)][(xb)a]\}x = [(xa)x]x = xx = 1.$ Hence

$x^{\#(a)} \leq x^{\#(b)} \vee a.$ Thus $x^{\#(a)} = x^{\#(b)} \vee a.$

5) Let $x,y \geq b$, then $x,y \in Q[b,1]$, so $z = \text{g.l.b}_{Q[b,1]}\{x,y\}$ exists. Next we show $z = \text{g.l.b}_Q\{x,y\}$. If $q \in Q$ with $q \leq x,y$, we want to show $q \leq z$. But $q \vee b \leq x,y$ implies $q \leq q \vee b \leq z$.

2.3 Orthoimplicative algebras.

2.3.1 Definition. (R. Godowski [3]).

Let $(A,\cdot,0,1)$ be an abstract algebra of type $(2,0,0)$. For all $a,b \in A$, we denote $a\cdot b$ by ab and $a0$ by a'. We call A an <u>orthoimplicative algebra</u> (OIA) if for all $a,b,c \in A$ the following hold:

G(1) $(ab)a = a$

G(2) $a(ba) = 1$

G(3) $(ab)b = (ba)a$

G(4) $a[c(ba)] = 1$

G(5) $0a = 1$

G(6) $a(ab)' = ab'$

G(7) $(a'b)((((ac)c)b)b) = 1$

G(8) $[(ac)b](c'b) = [(ac)b]b.$

2.3.2 <u>Lemma</u>. Let $(A,\cdot,0,1)$ be an orthoimplicative algebra. We define $a \leq b \equiv ab = 1$; then for all $a,b \in A$ we have the following:

1) $1a = a$

2) $a \leq (ab)b$

3) $[(ab)b]b = ab$

4) $ab = b$ iff $(ab)b = 1$ iff $ba = a$

5) $ab = 1$ iff $(ab)b = b$

6) $a(ab) = ab$

7) $aa' = a'$

8) $a'' = a$.

<u>Proof</u>.

The proofs follow from preceding definition.

2.3.3 <u>Lemma</u>. Let $(A,\cdot,0,1)$ be an orthoimplicative algebra. Then for any $a,b \in A$ the following are equivalent.

1) $a \leq b$

2) $a' = ab'$

3) $b' \leq a'$

4) $b = b'a$.

Proof.

$1 \to 2$: If $a \leq b$, then $ab = 1$. Thus $(ab)' = 0$ and so $a(ab)' = a'$. Hence by (G6) $a' = ab'$.

$2 \to 3$: $b'a' = b'(ab') = 1$. Hence $a' = ab'$.

$3 \to 4$ and $4 \to 1$ are equivalent to $1 \to 2$ and $2 \to 3$ respectively.

2.3.4 Definition. In an orthoimplicative algebra we write $a \perp b$ when $a \leq b'$.

2.3.5 Corollary. For any $a,b \in A$ we have $a \perp b$ iff $a' = ab$.

Proof.

$a \perp b$ iff $a \leq b'$ iff $a' = ab'' = ab$.

2.3.6 Theorem. Let $(A,\cdot,0,1)$ be an orthoimplicative algebra. Then

(i) '$\leq$' is a partial order relation,

(ii) The poset $(A,\leq)$ with the operation $a \mapsto a'$ is orthomodular,

(iii) If $a \mathrel{C} b$, then $ab = a' \vee b$.

Proof.

(i) 1) $aa = 1$ implies $a \le a$; hence '$\le$' is reflexive,

2) $a \le b$ and $b \le a$ imply $ab = ba = 1$. Thus by 2.3.3(5) $(ab)b = b$ and $(ba)a = a$. Therefore by (G3) $a = b$. Hence '$\le$' is anti-symmetric.

3) $a \le b$ and $b \le c$ imply $b = b'a$ and $c = c'b$ by 1.3.4. Hence $ac = a(c'b) = a[c'(b'a)] = a$ by (G4). Thus '$\le$' is transitive.

(ii) We have to show the following conditions.

1) $a'' = a$

2) $a \le b \Rightarrow b' \le a'$

3) If $a \le b'$, then the least upper bound $a \vee b$ exists in A.

4) $a \vee a' = 1$

5) $a \le b$ and $a \vee b' = 1$ imply $a = b$.

We have proved that conditions 1 and 2 held. Now we show the other conditions.

3) If $a \perp b$ then $a'b = (ab)b$. By 2.3.2 we have $a \le (ab)b$ and $b \le (ab)b$. Now if $a \le c$ and $b \le c$ then $\{[(ac)c]b\}b = (cb)b = c$, and then $(a'b)c = 1$ by (G7). So $(ab)b \le c$. Therefore $(ab)b = a \vee b$.

4) Since $a \perp a'$, $a \vee a' = (aa')a' = a'a' = 1$.

5) If $a \le b$, then $a \perp b'$. $a'b' = 1$ since $a \vee b' = 1$, hence $a' \le b'$, so $b \le a$. Therefore $a = b$.

(iii) Observe that if $x \perp y$, then by 2.3.5 $xy = x'$ and $x'y = x \vee y$. Now let $a \, C \, b$. Then there exist mutually orthogonal $a,b,c \in A$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. Hence $ab = (a_1 \vee c)(b_1 \vee c) = (a_1'c)(b_1'c) =$
$= [(a_1b_1)c](b_1'c)$. Therefore by (G8) $ab = [(a_1b_1)c]c =$
$= (a_1'c)c = a_1' = a' \vee b$.

2.3.7 Theorem. In an orthoimplicative algebra if we write $a^{\#(z)}$ for the orthocomplement of a in $[z,1]$, then $y \leq x \leq a$ implies $a^{\#(x)} = a^{\#(y)} \vee x$.

Proof.

First since $x \perp a^{\#(y)}$ in $[y,1]$, then $x \vee a^{\#(y)}$ exists. Now we need to show

(i) $a^{\#(x)} \geq a^{\#(y)} \vee x$ and (ii) $a^{\#(x)} \leq a^{\#(y)} \vee x$.

(i) Since $x \leq a^{\#(x)}$ we only need to show $a^{\#(y)} \leq a^{\#(x)}$ or $(a^{\#(x)})^{\#(y)} \leq a$ which is $(ax)y \leq a$ which is $[(ax)y]a = 1$. Since $a \, C \, x,y$ and $x \, C \, y$, we have

$[(ax)y]a = [(a^{\#(y)} \vee x)^{\#(y)} \vee y]^{\#(y)} \vee a = a \vee y^{\#(y)} =$

$= (a^{\#(y)})^{\#(y)} \vee y^{\#(y)} = a^{\#(y)}y^{\#(y)} = 1$ since

$a^{\#(y)} \leq y^{\#(y)}$. Therefore $a^{\#(y)} \leq a^{\#(x)}$, hence

$a^{\#(x)} \geq a^{\#(y)} \vee x$.

(ii) To show $a^{\#(x)} \leq a^{\#(y)} \vee x$ we need $a^{\#(x)}(a^{\#(y)} \vee x) = 1$

but $a^{\#(x)}(a^{\#(y)} \vee x) = (a^{\#(x)})^{\#(x)} \vee a^{\#(y)} \vee x = 1$.

Hence $a^{\#(x)} \leq a^{\#(y)} \vee x$. Thus $a^{\#(x)} = a^{\#(y)} \vee x$.

CHAPTER 3

GENERALIZED ORTHOMODULAR LATTICES

3.1 Definition. A generalized orthomodular lattice (GOML) is a lattice L with least element 0 on which there is defined an orthogonality relation "$\perp$" satisfying:

J1) $a \perp a$ iff $a = 0$

J2) $a \perp b$ implies $b \perp a$

J3) $a \perp b$ and $c \leq a$ imply $c \perp b$

J4) $a \perp b$ and $a \perp c$ imply $a \perp b \vee c$

J5) If $a,b \in L$ with $a \leq b$. There exist an $x \in L$ such that $a \perp x$ and $a \vee x = b$.

Remark. Notice that if $(L,\leq,',0,1)$ is an orthomodular lattice and we define $a \perp b$ iff $a \leq b'$, then $(L,\leq,0,1)$ is a generalized orthomodular lattice. Hence generalized orthomodular lattice generalize orthomodular lattices. The kernel of a congruence relation on an OML is a GOML but not necessarily a OML. This is one reason for the interest in GOMLs.

3.2 Lemma. In a generalized orthomodular lattice if $a \perp b$ and $a \vee b \perp c$, then $a \perp b \vee c$.

Proof.

We have $a \vee b \perp c$ and $a \leq a \vee b$. Therefore $a \perp c$ by (J3) and $a \perp b \vee c$ by (J4).

3.3 Theorem. Every generalized orthomodular lattice has the following properties:

Q1) If $b \varepsilon L$, then $L[0,b]$ is an orthomodular lattice.

Q2) If $x \leq a \leq b$ for all $x,a \varepsilon L[0,b]$, then

$x^{\#(a)} = x^{\#(b)} \wedge a$ where for all $y \varepsilon L[0,b]$, $x^{\#(y)}$ is the orthocomplement of x in $L[0,y]$.

Proof.

We first show that $L[0,b]$ is an orthocomplemented lattice. Therefore we must show there is a mapping $\#(b) : L[0,b] \to L[0,b]$ such that

(i) $L[0,b]$ is bounded

(ii) $a^{\#(b)\#(b)} = a \quad \forall a \varepsilon L[0,b]$

(iii) $a_1 \leq a_2$ implies $a_2^{\#(b)} \leq a_1^{\#(b)} \quad \forall a_1,a_2 \varepsilon L[0,b]$

(iv) $a \vee a^{\#(b)} = b$ and $a \wedge a^{\#(b)} = 0$.

(i) It is clear that every interval is bounded.

(ii) It is enough to show there is a unique $x \varepsilon L[0,b]$ such that $a \perp x$ and $a \vee x = b$ for all $a \varepsilon L[0,b]$. For suppose x_1,x_2 are such elements. So by (J5) there exist a $q \varepsilon L$ such that

$$q \perp x_2 \text{ and } (*) \quad q \vee x_2 = x_1 \vee x_2.$$

Since $a \perp x_1,x_2$, by (J4) $a \perp x_1 \vee x_2 = q \vee x_2$. Now $q \perp x_2$ and $a \perp q \vee x_2$ implies $q \perp a \vee x_2 = b$ by (3.2).

So $q \perp b$ and $q \leq b$ implies $q \perp q$ which mean $q = 0$. Thus by (*) $x_2 = x_1 \vee x_2$ so $x_1 \leq x_2$ and by symmetry $x_2 \leq x_2$. Therefore $x_1 = x_2$ and hence for all $a \in L[0,b]$ there exist a unique $x \in L[0,b]$ such that $x \perp a$ and $x \vee a = b$. Now define $a^{\#(b)}$ to be the above unique $x \in L[0,b]$.

(iii) Let $a_1, a_2 \in L[0,b]$ with $a_1 \leq a_2$. Since $a_1 \perp a_1^{\#(b)}$ and $a_2 \perp a_2^{\#(b)}$, $a_1 \perp a_2^{\#(b)}$ and then $a_1 \perp a_1^{\#(b)} \vee a_2^{\#(b)}$. But $a_1 \vee (a_1^{\#(b)} \vee a_2^{\#(b)}) \geq a_1 \vee a_1^{\#(b)} = b$ so by (ii) $a_1^{\#(b)} \vee a_2^{\#(b)} = a_1^{\#(b)}$. Hence $a_1^{\#(b)} \geq a_2^{\#(b)}$.

(iv) First note that by (J3) $a \wedge a^{\#(b)} \perp a, a^{\#(b)}$ so $a \wedge a^{\#(b)} \perp a \vee a^{\#(b)} = b$ and since $a \wedge a^{\#(b)} \leq b$, $a \wedge a^{\#(b)} \perp a \wedge a^{\#(b)}$. Thus $a \wedge a^{\#(b)} = 0$. And this completes the proof that $L[0,b]$ is an ortho-complemented lattice.

Now before showing $L[0,b]$ is an OML we prove (Q_2) using the fact that in $L[0,b]$ we have

$$a_1 \leq a_2^{\#(b)} \quad \text{iff} \quad a_1 \perp a_2.$$

($\Rightarrow$) Let $a_1 \leq a_2^{\#(b)}$, so by (J3) $a_1 \perp a_2$.

($\Leftarrow$) Let $a_1 \perp a_2$, so by (J2) $a_2 \perp a_1$ and then $a_2 \perp a_1 \vee a_2^{\#(b)}$.

Since $a_2 \vee (a_1 \vee a_2^{\#(b)}) \leq a_2 \vee a_2^{\#(b)} = b$, $a_2^{\#(b)} = a_1 \vee a_2^{\#(b)}$ and thus $a_1 \leq a_2^{\#(b)}$.

Now we show (Q2).

(α) Since $x^{\#(a)} \perp x = (x^{\#(b)})^{\#(b)}$, $x^{\#(a)} \leq x^{\#(b)}$ thus $x^{\#(a)} \leq x^{\#(b)} \wedge a$.

(β) Since $x^{\#(b)} \wedge a \leq x^{\#(b)}$, $x^{\#(b)} \wedge a \perp x = (x^{\#(a)})^{\#(a)}$ thus $x^{\#(b)} \wedge a \leq x^{\#(a)}$. Hence $x^{\#(b)} \wedge a = x^{\#(a)}$.

Now to show (Q1). We already know that $L[0,b]$ is an OCL, so we need to show the orthomodular identity holds, i.e., $x,y \in L[0,b]$, $x \leq y \Rightarrow y = x \vee (y \wedge x^{\#(b)})$ but by (Q2) $x^{\#(y)} = y \wedge x^{\#(b)}$ and $y = x \vee x^{\#(y)}$ therefore $y = x \vee (y \wedge x^{\#(b)})$.

Let L be a generalized orthomodular lattice and for each subset M of L, let $M^{\perp} = \{x \in L : x \perp y, \forall y \in M\}$. If $x_1, x_2 \in M^{\perp}$ then $x_1 \vee x_2 \in M^{\perp}$ and also for $x_3 \leq x_1$ we have $x_3 \in M^{\perp}$ therefore $M^{\perp}$ is an ideal of L. Now let $I(L)$ denote the lattice of ideals of L, and J_x denote the principle ideal generated by x and finally let $\mathcal{M}$ be the set of those ideals I of L such that $I = J_x$ or $I = J_x^{\perp}$ for some $x \in L$.

3.4 Lemma. If $I \in \mathcal{M}$ then $I = (I^{\perp})^{\perp}$.

Proof.

(i) If $I = J_x^{\perp}$ then clearly $I = (I^{\perp})^{\perp}$.

(ii) If $I = J_x$ then let $a \in I^{\perp\perp}$ and work in $L[0, a \vee x]$ which by (Q1) is OML, so we have $x^{\perp} \in L$ with $x \vee x^{\perp} = a \vee x$ with $x \perp x^{\perp}$ and $a \perp x^{\perp}$. This implies $a \leq x$ and hence $(I^{\perp})^{\perp} \subseteq I$ and $I \subseteq (I^{\perp})^{\perp}$ so this implies $(I^{\perp})^{\perp} = I$.

3.5 <u>Theorem</u>. M is a sublattice of $I(L)$.

<u>Proof</u>.

Given $I_1, I_2 \in M$ we must show $I_1 \vee I_2 \in M$ and $I_1 \wedge I_2 \in M$ where $I_1 \vee I_2 = \{z : z \leq x \vee y, x \in I_1, y \in I_2\}$.

<u>Case 1</u>: $I_1 = J_x$, $I_2 = J_y$.

Then $I_1 \vee I_2 = J_x \vee J_y = J_{x \vee y} \in M$ and $I_1 \wedge I_2 = J_{x \wedge y} \in M$.

<u>Case 2</u>: $I_1 = J_x^{\perp}$ and $I_2 = J_y^{\perp}$.

(α) We want to show $I_1 \vee I_2 = J^{\perp}_{x \wedge y}$. First note that $(I_1 \vee I_2)^{\perp} = I_1^{\perp} \wedge I_2^{\perp} = J_x \wedge J_y = J_{x \wedge y}$. Now let $a \perp x \wedge y$ and work in $L[0, a \vee x \vee y]$ which is an OML. We have $a \perp x \wedge y$ and $a \leq a \vee x \vee y$ hence $a \leq (x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$, where $(x \wedge y)^{\perp}$ is the orthocomplement of $x \wedge y$ in $L[0, a \vee x \vee y]$. Therefore $I_1 \vee I_2 = (I_1 \vee I_2)^{\perp\perp} = J^{\perp}_{x \wedge y} \in M$.

(β) Note that $a \in I_1 \wedge I_2$ iff $a \perp x$ and $a \perp y$ iff $a \perp x \vee y$. Thus $I_1 \wedge I_2 = J^{\perp}_{x \vee y} \in M$.

<u>Case 3</u>: $I_1 = J_x$ and $I_2 = J_y^{\perp}$.

(α) Let $x \vee y = y \vee y^{\perp}$ with $y \perp y^{\perp}$ where $y^{\perp}$ is the orthocomplement of y in $L[0, x \vee y]$.

Claim: $I_1 \wedge I_2 = J_{x \wedge y^{\perp}}$.

Let $a \in I_1 \wedge I_2$ so $a \leq x$ and $a \perp y$. Thus $a \leq x \vee y$, $a \perp y$ puts $a \leq y^{\perp}$ and hence $a \leq x \wedge y^{\perp}$. If $a \leq x \wedge y^{\perp}$ and $a \in I_1 \wedge I_2$ then $I_1 \wedge I_2 = J_{x \wedge y^{\perp}} \in M$.

(β) Note that $(I_1 \vee I_2)^{\perp} = I_1^{\perp} \wedge I_2^{\perp} = J_x^{\perp} \wedge J_y$.

Let $x \vee y = x \vee x^{\perp}$ with $x \perp x^{\perp}$ and by interchanging the role of x and y in (α) we see that $J_y \wedge J_x^{\perp} = J_{y \wedge x^{\perp}}$. Thus $(I_1 \vee I_2)^{\perp} = J_{y \wedge x^{\perp}}$.

Claim: $I_1 \vee I_2 = (I_1 \wedge I_2)^{\perp\perp} = J_{y \wedge x^{\perp}}^{\perp}$.

Let $a \perp y \wedge x^{\perp}$. Working in $L[0, a \vee x \vee y]$, if we let $x^{\#}$ denote the orthocomplement of x, then by (Q2) we have $x^{\perp} = x^{\#} \wedge (x \vee y)$. Now $a \perp x^{\perp} \wedge y$ implies $a \leq (x^{\perp} \wedge y)^{\#} = x^{\perp\#} \vee y^{\#} = ((x \vee y) \wedge x^{\#})^{\#} \vee y^{\#} = x \vee y^{\#}$. This puts $a \in I_1 \vee I_2 = J_x \vee J_x^{\perp}$ and establishes $(I_1 \vee I_2)^{\perp\perp} = I_1 \vee I_2 \in M$.

3.6 Theorem. M is an orthomodular lattice.

Proof.

By definition of M, $I \in M \Rightarrow I^{\perp} \in M$ and by the Lemma, $I \to I^{\perp}$ is an involution on M. Since $I \cap I^{\perp} = (0)$, this involution is actually an orthocomplementation. Now we have to show $M(I, I^{\perp})$ holds for all $I \in M$. This will be established by showing that $M(J_x, J_x^{\perp})$ and $M(J_x^{\perp}, J_x)$ both hold for each $x \in L$; an OCL is an OML if orthogonal pairs are modular pairs [6, p. 100].

[$M(J_x^{\perp}, J_x)$ is equivalent to $I \leq J_x$ implies $(I \vee J_x^{\perp}) \wedge J_x = I$].

Claim 1: $M(J_x^{\perp}, J_x)$ holds.

Let $I \leq J_x$, then if $a \in (I \vee J_x^{\perp}) \wedge J_x$, $a \leq b \vee c$ with $b \in I$ and $c \perp x$. Since $a \leq x$ we have $a \leq (b \vee c) \wedge x =$

$= b \vee (c \wedge x) = b$. Thus $(I \vee J_x^{\perp}) \wedge J_x = I$ and therefore $M(J_x^{\perp}, J_x)$ holds.

Claim 2: $M(J_x, J_x^{\perp})$ holds.

Let $I \leq J_x^{\perp}$. If $a \in (I \vee J_x) \wedge J_x^{\perp}$, then $a \perp x$ and $a \leq b \vee x$ with $b \in I \subseteq J_x^{\perp}$; hence $a \vee b \perp x$ and $a \leq (b \vee x) \wedge (a \vee b) = b \vee (x \wedge (a \vee b)) = b \in I$. Thus $M(J_x, J_x^{\perp})$ holds. Therefore $\mathcal{M}$ is an orthomodular lattice.

3.7 Definition. An ideal I of an orthomodular lattice is said to be prime if $a_{,} b \in I$ implies $a \in I$ or $b \in I$.

3.8 Lemma. Let J be a nonempty subset of an orthomodular lattice K such that: (α) $1 \notin J$ and (β) J is closed under the formation of finite suprema. Then J is a prime ideal of K iff for each $x \in K$ either $x \in J$ or $x' \in J$. A proper prime ideal is maximal.

Proof.

If J is a prime ideal of K, then $x \wedge x' = 0 \in J$ forces $x \in J$ or $x' \in J$. Suppose conversely that for each $x \in K$, either $x \in J$ or $x' \in J$. Then $1 \notin J$ implies $0 \in J$. If $a \in J$ and $b \leq a$ we must have $b \in J$, since otherwise $b' \in J$ would imply that $1 = a \vee b' \in J$, a contradiction. This shows that J is an ideal of L. Now if $a \wedge b \in J$, $a \notin J$, $b \notin J$, then $a' \in J$, $b' \in J$. Once again this forces

$$1 = (a \wedge b) \vee a' \vee b' \in J, \text{ a contradiction.}$$

We conclude that J is a prime ideal of K. An easy argument that J is maximal is [6, p. 88].

Returning now to our situation we have

3.9 Theorem. L is isomorphic to a prime ideal of M.

Proof.

The mapping $x \to J_x$ is clearly an isomorphism of L onto a sublattice of M. Also, by definition of M, either $I = J_x$ or $I^{\perp} = J_x$ for each $I \in M$. By Lemma 4, $\{J_x | x \in L\}$ is a prime ideal of M.

Notice that the mapping $x \to J_x$ of L into M preserves arbitrary suprema and infima whenever they exist in L. The assertion concerning infima is obvious, so let $x = \vee_\alpha x_\alpha$ exist in L. Then $J_x \supseteq J_{x_\alpha}$ holds in M for each index α. If $I \in M$ and $I \geq$ all J_{x_α}, then $I \cap J_x \geq$ all J_{x_α}. But $I \cap J_x \leq J_x$ forces $I \cap J_x = J_y$ for some $y \in L$. But now $J_y \geq J_{x_\alpha}$ implies $y \geq x_\alpha$, and since this holds for every choice of α, we have $y \geq x = \vee_\alpha x_\alpha$. Thus $I \cap J_x = J_x \leq I$, so $J_x = \vee_\alpha J_{x_\alpha}$ in M.

3.10 Corollary. M is an orthocomplemented modular lattice if and only if L is modular.

Proof.

M is a sublattice of $I(L)$ and L is an ideal of M. The Corollary follows from [2, pp. 13, 113].

3.11 <u>Theorem</u>. Let L be a lattice such that for each $x \in L$. $L[0,x]$ with the induced order and orthocomplementation $\#(x) : L[0,x] \to L[0,x]$ is an orthomodular lattice. Suppose also that if $a,x,y \in L$ and $a \leq x \leq y$, then $a^{\#(x)} = a^{\#(y)} \wedge x$. Then L is a generalized orthomodular lattice if $a \perp b$ is defined by $a \leq b^{\#(a \vee b)}$.

<u>Proof</u>.

To show L is GOML we have to show J1-5 holds.

(J1) $a \perp a \Rightarrow a \leq a^{\#(a)} = 0 \Rightarrow a = 0$.

(J2) $a \perp b \Rightarrow a \leq b^{\#(a \vee b)} \Rightarrow a^{\#(a \vee b)} \geq b$, since $L[0,a \vee b]$ is an OML.

(J3) $a \perp b$ and $a_1 \leq a$. So $a_1 \leq a \leq b^{\#(a \vee b)}$ we want $a_1 \leq b^{\#(a_1 \vee b)}$ since $b \leq a_1 \vee b \leq a \vee b$, $b^{\#(a_1 \vee b)} =$ $= b^{\#(a \vee b)} \wedge (a_1 \vee b)$. We have $a_1 \leq b^{\#(a \vee b)}$, $a_1 \vee b$ so $a_1 \leq b^{\#(a \vee b)} \wedge (a_1 \vee b) = b^{\#(a_1 \vee b)}$ hence $a_1 \perp b$.

(J4) $a \perp b$ and $a \perp c \Rightarrow a \leq b^{\#(a \vee b)}$, $c^{\#(a \vee c)}$ we have $b \leq a \vee b \leq a \vee b \vee c$. Hence by (Q2) we want $a \perp b \vee c$.

$b^{\#(a \vee b)} = b^{\#(a \vee b \vee c)} \wedge (a \vee b) \leq b^{\#(a \vee b \vee c)}$ and

hence $a \leq b^{\#(a \vee b)} \leq b^{\#(a \vee b \vee c)}$. Similarly since

$c \leq a \vee c \leq a \vee b \vee c$, we have $a \leq c^{\#(a \vee c)} \leq c^{\#(a \vee b \vee c)}$.

Therefore $a \leq b^{\#(a \vee b \vee c)} \wedge c^{\#(a \vee b \vee c)} =$

$(b \vee c)^{\#(a \vee b \vee c)}$ hence $a \perp (b \vee c)$.

(J5) Let $a, b \in L$ with $a \leq b$. We want to show that there exist $x \in L$ with $a \perp x$ such that $a \vee x = b$. By G1, $L[0,b]$ is an OML, and so we have $a \vee a^{\#(b)} = b$. Set $x = a^{\#(b)}$. Then $a \vee x = b$ so it remains to show that $a \perp x$. Notice that $a \leq a = (a^{\#(b)})^{\#(b)} =$ $= x^{x(b)} = x^{\#(a \vee x)}$ so $a \leq x^{\#(a \vee x)}$. Thus $a \perp x$.

Hence L is a generalized orthomodular lattice.

3.12 __Lemma__. In an orthomodular lattice, define $a \cdot b$ as follows: $ab = a' \vee (a \wedge b)$. If $x \leq a,b$ and $\#$ is relativized orthocomplementation in $[x,1]$, that is $y^{\#} = y' \vee x$. Then $ab = a^{\#} \vee (a \wedge b)$.

__Proof__.

$$a^{\#} \vee (a \wedge b) = (a' \vee x) \vee (a \wedge b) = a' \vee (a \wedge b) = ab.$$

3.13 __Remark__. Let $(P, \leq)$ be a poset. Then the dual of P written δP is the poset formed by P with the partial ordering $\preccurlyeq$ defined by $x \preccurlyeq y$ iff $y \leq x$. Note that the dual of a lattice is a lattice and the dual of an orthomodular lattice is an orthomodular lattice.

3.14 __Theorem__. Let L be the dual of a generalized orthomodular lattice. Then

(δQ1) If $b \in L$ then $L[b,1]$ is an orthomodular lattice.

(δQ2) If $x \geq a \geq b$ for all $x,a \in L[b,1]$. Then $x^{\#(a)} = x^{\#(b)} \vee a$, where for all $y \in L[b,1]$ $x^{\#(y)}$ is the orthocomplement of x in $L[y,1]$.

Proof.

The proof is dual to the proof of Theorem 3.3.

3.15 Theorem. The dual of a generalized orthomodular lattice is a quasi implication algebra with $ab :=_{def} a^{\#(a \wedge b)}$ where $a^{\#(a \wedge b)} = a' \vee (a \wedge b)$ and $'$ is the orthocomplementation on $[x,1]$, for any $x \leq a,b$.

Proof.

Let L be a δ GOML and let $a,b,c \in L$. By 4.10 for any $x \in L$, $[x,1]$ is an OML. For $y \in [x,1]$ let $y^{\#(x)}$ be the orthocomplement of y in $[x,1]$. Then we have to show the following

(i) $(ab)a = a$

(ii) $(ab)(ac) = (ba)(bc)$

(iii) $[(ab)(ba)]a = [(ba)(ab)]b$.

(i) $(ab)a = a^{\#(a \wedge b)}a = (a^{\#(a \wedge b)})^{\#(a^{\#(a \wedge b)} \wedge a)} =$

$= (a^{\#(a \wedge b)})^{\#(a \wedge b)} = a.$

(ii) $(ab)(ac) = (ab)^{\#(ab \wedge ac)}$. But

$ab \wedge ac = a^{\#(a \wedge b)} \wedge a^{\#(a \wedge c)} = [a' \vee (a \wedge b)] \wedge [a' \vee (a \wedge c)]$

$= a' \vee (a \wedge b \wedge c) = a'$

where $'$ is the orthocomplementation of $[a \quad b \quad c,1]$.

So $(ab)(ac) = (ab)^{\#(a')} = (ab)' \vee a' = [a' \vee (a \wedge b)]' \vee a'$

$= a' \vee b'$ and by symmetry $(ba)(bc) = a' \vee b'$ hence

$(ab)(ac) = (ba)(bc)$.

(iii) $[(ab)(ba)]a = [(ab)(ba)]^{\#[(ab)(ba) \wedge a]}$. But

$(ab)(ba) = (ab)^{\#(ab \wedge ba)} = (a^{\#(a \wedge b)})^{\#(a^{\#(a \wedge b)} \wedge b^{\#(a \wedge b)})}$

and $a^{\#(a \wedge b)} \wedge b^{\#(a \wedge b)} = [a' \vee (a \wedge b)] \wedge [b' \vee (a \wedge b)] =$

$= (a \wedge b) \vee (a' \wedge b')$. Therefore $(ab)(ba) =$

$= [a' \vee (a \wedge b)]' \vee (a \wedge b) \vee (a' \wedge b') = [a \wedge (a \wedge b)'] \vee$

$\vee (a \wedge b) \vee (a' \wedge b') = a \vee (a' \wedge b')$. Hence

$[(ab)(ba)]a = [a \vee (a' \wedge b')]^{\#[(a \vee (a' \wedge b')) \wedge a]} =$

$= [a \vee (a' \wedge b')]^{\#(a)} = [a \vee (a' \wedge b')]' \vee a =$

$= [a' \wedge (a \vee b)] \vee a = a \vee b$ and by symmetry

$[(ba)(ab)]b = a \vee b$. Thus $[(ab)(ba)]a =$

$= [(ba)(ab)]b$. Therefore every dual GOML is a QIA.

As a consequence of Theorem 2.2.10 and the preceding Theorem, we see that quasi implication algebras overgeneralize dual generalized orthomodular lattices only to the extent that two elements in a QIA which are not bounded below need not have an infimum.

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IMPLICATION ALGEBRAS

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Abstract

An implication algebra, introduced by J. C. Abbott [1], is a pair $\langle I,\cdot\rangle$ where I is a set and for all $a,b,c \in I$ $\cdot : I\times I \to I$ satisfies

A1) $(a\cdot b)\cdot a = a$

A2) $(a\cdot b)\cdot b = (b\cdot a)\cdot a$

A3) $a\cdot(b\cdot c) = b\cdot(a\cdot c)$.

If one begins with a Boolean algebra $\langle B,\vee,\wedge,',0,1\rangle$ and defines $a\cdot b$ to be $a'\vee b$ for all $a,b \in B$ then $\langle B,\cdot\rangle$ is an implication algebra.

In the other direction, we show that every implication algebra $\langle I,\cdot\rangle$ possesses a distinguished element $1 \in I$ such that $\forall\ a \in I$, $aa = a1 = 1$ and $1a = a$. Thus, by defining $a \leq b \Leftrightarrow ab = 1$, we obtain a poset $(I,\leq)$ with distinguished element 1 such that $[a,1] = \{x \in I : a \leq x \leq 1\}$ is a Boolean algebra.

In Chapter 2, we discuss quasi implication algebras, which generalize orthomodular lattices as the above implication algebras generalize Boolean algebras. In particular, we show that each implication algebra is a quasi implication algebra, and each quasi implication algebra satisfying the exchange property $a\cdot(b\cdot c) = b\cdot(a\cdot c)$ $\forall a,b,c \in I$ is an implication algebra. Moreover, if one begins with an orthomodular lattice (rather than a Boolean algebra as above) and defines $a\cdot b$ to be $a'\vee(a\wedge b)$, then $\langle I,\cdot\rangle$ is a quasi implication algebra.

Chapter 2 concludes with a discussion of orthoimplicative algebras, and their relation to orthomodular partially ordered sets.

Chapter 3 continues with the study of generalized orthomodular lattices introduced by Janowitz in [2]. These are lattices L with a least element 0 and binary relation $\perp \subseteq L \times L$ called <u>orthogonality</u> and satisfying, for all $a,b,c \in L$;

J1) $a \perp a \iff a = 0$

J2) $a \perp b \implies b \perp a$

J3) $a \perp b$ and $c \leq a \implies c \perp b$

J4) $a \perp b$ and $a \perp c \implies a \perp (b \vee c)$

J5) $a,b \in L, a \leq b \implies$ there is an $x \in L$ such that $a \perp x$ and $a \vee x = b$

where we have written $a \perp b$ for $\langle a,b\rangle \in \perp$. We show that there is in fact a unique element satisfying the conclusion of J5, and denote it by $a^{\#(b)}$. If we let $[0,b]$ denote $\{x \in L : 0 \leq x \leq b\}$, it then follows that the mapping $a \to a^{\#(b)}$ is an orthocomplementation on $[0,b]$ making $[0,b]$ an orthomodular lattice. In addition, the following "relativization property" holds: if $x \leq a \leq b$ then $x^{\#(a)} = x^{\#(b)} \wedge a$. Conversely, we have the following

<u>Theorem</u>: Let L be a lattice. Suppose that for each $x \in L$, $[0,x]$ is an orthomodular lattice with orthocomplementation $\#(x) : [0,x] \to [0,x]$. Suppose further that the relativization property holds. Define $\perp \subseteq L \times L$ by $\langle a,b\rangle \in \perp$ iff $a \leq b^{\#(a \vee b)}$. Then $\langle L,0,\perp\rangle$ is a generalized orthomodular lattice.

Chapter 3 continues with a discussion of the embedding of a GOML in an orthomodular lattice, and concludes with a proof of the following

Theorem: Let L be a dual generalized orthomodular lattice. Let $\#(x)$ denote the orthocomplementation on $[x,1]$ and define $a \cdot b$ to be $a^{\#(a \wedge b)}$. Then $\langle L, \cdot \rangle$ is a quasi implication algebra.